Curvilinear Coordinates in Geometric Algebra

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Contents

Let $U \subseteq R^n$ be open. Let $r_1, r_2, ...r_n$ be the coordinates of a point $\vec{r}$ with respect to an orthonormal basis so that $\vec{r} = r_i \vec{e}_i$. And let $w_1, w_2...w_n$ be the coordinates of the same point $\vec{r}$ in a curvilinear coordinate system defined in U.

Example:

$$x = r\cos\varphi \qquad y = r\sin\varphi$$

$$r = \sqrt{x^2+y^2} \qquad \tan\varphi = \frac{y}{x}$$

We only allow coordinates $\{w_k\}$ where the coordinate change $\{w_k\} \to \{r_i\}$ is continuously differentiable and has an invertible diffential at any point in U. Therefore the partial derivatives $\partial r_i / \partial w_k$ and $\partial w_i / \partial r_k$ exist and are continuous.

Let's check the existence of $\partial r_i / \partial w_k$ for polar coordinates.

$$\frac{\partial\left(x\left(r,\varphi\right)\right)}{\partial r} = \cos\varphi \qquad \frac{\partial\left(x\left(r,\varphi\right)\right)}{\partial\varphi} = -r\sin\varphi$$

$$\frac{\partial\left(y\left(r,\varphi\right)\right)}{\partial r} = \sin\varphi \qquad \frac{\partial\left(y\left(r,\varphi\right)\right)}{\partial\varphi} = r\cos\varphi$$

Let's check the existence of $\partial w_i / \partial r_k$ for polar coordinates.

$$\frac{\partial\left(r\left(x,y\right)\right)}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \qquad \frac{\partial\left(r\left(x,y\right)\right)}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial\varphi\left(x,y\right)}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial\varphi\left(x,y\right)}{\partial y} = \frac{x}{x^2+y^2}$$

A curvilinear coordinate system $\{w_k\}$ determines two bases at each point in U, $\{\vec{w}_k\}$ and $\left\{\vec{w}^j\right\}$ in which j is a superscript, not an exponent. They are defined by

$$\vec{w}_k = \frac{\partial\vec{r}}{\partial w_k} = \frac{\partial\left(r_i\vec{e}_i\right)}{\partial w_k} = \frac{\partial r_i}{\partial w_k}\vec{e}_i$$

and

$$\vec{w}^j = \nabla w_j = \vec{e}_i \partial_i w_j = \frac{\partial w_j}{\partial r_i} \vec{e}_i$$

Example: We define

$$\vec{r} = r \cos\varphi \vec{e}_x + r \sin\varphi \vec{e}_y$$

What are the basis vectors $\vec{w}_r$ and $\vec{w}_\varphi$ for the polar coordinates definition? Generally we have

$$\boxed{\vec{w}_k = \frac{\partial r_i}{\partial w_k} \vec{e}_i} \qquad \text{Curvilinear base vectors} \tag{1}$$

and thus in our case

$$\vec{w}_r = \frac{\partial x\,(r, \varphi)}{\partial r} \vec{e}_x + \frac{\partial y\,(r, \varphi)}{\partial r} \vec{e}_y = \cos\varphi \vec{e}_x + \sin\varphi \vec{e}_y$$

$$\vec{w}_\varphi = \frac{\partial x\,(r, \varphi)}{\partial \varphi} \vec{e}_x + \frac{\partial y\,(r, \varphi)}{\partial \varphi} \vec{e}_y = -r \sin\varphi \vec{e}_x + r \cos\varphi \vec{e}_y$$

Let's check whether these curvilinear base vectors *are orthogonal.* They are orthogonal if the scalar product turns out to be zero.

$$\begin{aligned}
\vec{w}_r \cdot \vec{w}_\varphi &= (\cos\varphi \vec{e}_x + \sin\varphi \vec{e}_y) \cdot (-r \sin\varphi \vec{e}_x + r \cos\varphi \vec{e}_y) \\
\vec{w}_r \cdot \vec{w}_\varphi &= (\cos\varphi \vec{e}_x) \cdot (-r \sin\varphi \vec{e}_x + r \cos\varphi \vec{e}_y) + (\sin\varphi \vec{e}_y) \cdot (-r \sin\varphi \vec{e}_x + r \cos\varphi \vec{e}_y) \\
\vec{w}_r \cdot \vec{w}_\varphi &= (\cos\varphi \vec{e}_x) \cdot (-r \sin\varphi \vec{e}_x) + (\sin\varphi \vec{e}_y) \cdot (r \cos\varphi \vec{e}_y) \\
\vec{w}_r \cdot \vec{w}_\varphi &= -r \sin\varphi \cos\varphi \vec{e}_x + r \sin\varphi \cos\varphi \\
\vec{w}_r \cdot \vec{w}_\varphi &= 0
\end{aligned}$$

What about the magnitude of these base vectors?

$$\begin{aligned}
|\vec{w}_r| &= 1 \\
|\vec{w}_\varphi| &= r
\end{aligned}$$

Obviosuly curvilinear base vectors are nort necessarily orthonormal.

Let's now examine the reciprocal base vectors $\vec{w}^r$ and $\vec{w}^\varphi$. Generally we have

$$\boxed{\vec{w}^j = \frac{\partial w_j}{\partial r_i} \vec{e}_i} \qquad \text{Reciprocal base vectors} \tag{2}$$

and thus in our case

$$
\begin{aligned}
\vec{w}^r &= \frac{\partial r\left(x,y\right)}{\partial x}\vec{e}_x + \frac{\partial r\left(x,y\right)}{\partial y}y \\
\vec{w}^r &= \frac{x}{\sqrt{x^2+y^2}}\vec{e}_x + \frac{y}{\sqrt{x^2+y^2}}\vec{e}_y \\
\vec{w}^r &= \frac{x}{r}\vec{e}_x + \frac{y}{r}\vec{e}_y
\end{aligned}
$$

$$
\begin{aligned}
\vec{w}^\varphi &= \frac{\partial \varphi\left(x,y\right)}{\partial x}\vec{e}_x + \frac{\partial \varphi\left(x,y\right)}{\partial y}y \\
\vec{w}^\varphi &= \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial x}\vec{e}_x + \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial y}\vec{e}_y
\end{aligned}
$$

$$
\vec{w}^\varphi = -\frac{y}{x^2+y^2}\vec{e}_x + \frac{x}{x^2+y^2}\vec{e}_y = -\frac{y}{r^2}\vec{e}_x + \frac{x}{r^2}\vec{e}_y
$$

What about the magnitude of these reciprocal base vectors?

$$
\begin{aligned}
\left|\vec{w}^r\right| &= \sqrt{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2} = 1 \\
\left|\vec{w}^\varphi\right| &= \sqrt{\left(\frac{y}{r^2}\right)^2 + \left(\frac{x}{r^2}\right)^2} = \frac{1}{r}
\end{aligned}
$$

We obviously have

$$
\begin{aligned}
\left|\vec{w}_r\right|\left|\vec{w}^r\right| &= 1 \\
\left|\vec{w}_\varphi\right|\left|\vec{w}^\varphi\right| &= 1
\end{aligned}
$$

In general neither $\{\vec{w}_k\}$ nor $\left\{\vec{w}^j\right\}$ is an orthonormal basis but we have

$$
\begin{aligned}
\vec{w}_k \cdot \vec{w}^j &= \left(\frac{\partial r_l}{\partial w_k}\vec{e}_l\right)\cdot\left(\frac{\partial w_j}{\partial r_i}\vec{e}_i\right) \\
&= \left(\frac{\partial r_i}{\partial w_k}\vec{e}_i\right)\cdot\left(\frac{\partial w_j}{\partial r_i}\vec{e}_i\right) \\
&= \frac{\partial r_i}{\partial w_k}\frac{\partial w_j}{\partial r_i} \\
&= \frac{\partial w_j}{\partial w_k} \\
&= \begin{cases} 1 & : for j = k \\ 0 & : for j \neq k \end{cases}
\end{aligned}
\tag{3}
$$

The reciprocal base vector $\vec{w}^j$ is obviously parallel to $\vec{w}_j$. We thus have $\vec{w}^j \wedge \vec{w}_j = 0$ and therefore $\vec{w}^j \cdot \vec{w}_j = \vec{w}^j\vec{w}_j = 1$. This implies that $\vec{w}^j$ is the inverse of $\vec{w}_j$.

$$\boxed{\vec{w}^j = \vec{w}_j^{-1} = \frac{1}{|\vec{w}_j|^2}\vec{w}_j} \tag{4}$$

This is super useful since $\vec{w}_j$ is often easier to calculate than $\vec{w}^j$. Unlike $\vec{e}_i$ the bases $\vec{w}_k$ and $\vec{w}^j$ can vary from point to point. Thus they cannot be treated as constant when differentiating.

Let's consider

$$\nabla \wedge \vec{w}^j = (\vec{e}_k \partial_k) \wedge \left(\frac{\partial w_j}{\partial r_i}\vec{e}_i\right) = \left(\vec{e}_k \frac{\partial}{\partial_{r_k}}\right) \wedge \left(\frac{\partial w_j}{\partial r_i}\vec{e}_i\right)$$

For $k = i$ we have a wedge prooduct of parallel vectors and thus null. For $i \neq k$ we get pairs like

$$\left(\vec{e}_k \frac{\partial}{\partial_{r_k}}\right) \wedge \left(\frac{\partial w_j}{\partial r_i}\vec{e}_i\right) + \left(\vec{e}_i \frac{\partial}{\partial_{r_i}}\right) \wedge \left(\frac{\partial w_j}{\partial r_k}\vec{e}_k\right) = \left(\frac{\partial}{\partial_{r_k}} \frac{\partial w_j}{\partial r_i}\vec{e}_k\right) \wedge \vec{e}_i + \left(\frac{\partial}{\partial_{r_i}} \frac{\partial w_j}{\partial r_k}\vec{e}_i\right) \wedge \vec{e}_k = \left(\left(\frac{\partial^2 w_j}{\partial r_i \partial r_k} - \frac{\partial^2 w_j}{\partial r_i \partial r_k}\right) \vec{e}_k\right.$$

and thus

$$\boxed{\nabla \wedge \vec{w}^j = 0}$$

Theorem: Let $\begin{pmatrix} w_1 & w_2 & w_n \end{pmatrix}$ be a curvilinear coordinate system in U then the gradient in curvilinear coordinates is given by

$$\boxed{\nabla = \vec{w}^j \partial_{w_j}} \tag{5}$$

Substituting Eq. 2 into this equation gives

$$\begin{aligned}
\nabla &= \vec{w}^j \partial_{w_j} \\
\nabla &= \frac{\partial w_j}{\partial r_i}\vec{e}_i \frac{\partial}{\partial_{w_j}} \\
\nabla &= \vec{e}_i \frac{\partial}{\partial r_i} \\
\nabla &= \vec{e}_i \partial_i
\end{aligned}$$

and thus the gradient in cartesian coordinates.

$$\nabla = \vec{w}^j \partial_{w_j}$$

With Eq. 4 and Eq. 1 and Eq. 5

$$\vec{w}^j = \frac{1}{|\vec{w}_j|^2}\vec{w}_j \qquad \vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad \nabla = \vec{w}^j \partial_{w_j}$$

we get

$$\boxed{\nabla = \frac{1}{\left|\vec{w}_j\right|^2}\vec{w}_j\partial_{w_j}} \qquad \text{Gradient in curvilinear coordinates} \tag{6}$$

We also have

$$\vec{w}_j = \left|\vec{w}_j\right|\vec{e}_{w_j} \qquad\qquad \vec{e}_{w_j} = \frac{1}{\left|\vec{w}_j\right|}\vec{w}_j$$

where $\vec{e}_{w_j}$ is the curvilinear unit vectors. This lets us write Eq. 6 like this

$$\boxed{\nabla = \frac{1}{\left|\vec{w}_j\right|}\vec{e}_{w_j}\partial_{w_j}} \qquad \text{Gradient in curvilinear coordinates} \tag{7}$$

1 Spherical coordinates

Let a curvilinear coordinate system be defined by the expression

$$\vec{r}\,(\alpha,\beta,r) = \begin{pmatrix} r\cos\alpha\sin\beta \\ r\sin\alpha\sin\beta \\ r\cos\beta \end{pmatrix}$$

Our aim is to calculate Eq. 6 for this coordinate system ($w_1 = \alpha$, $w_2 = \beta$, $w_3 = r$). We first determine the three

$$\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad\qquad \vec{e}_{w_j} = \frac{1}{\left|\vec{w}_j\right|}\vec{w}_j$$

and their magnitudes.

$$\begin{aligned}
\vec{w}_\alpha &= \frac{\partial x}{\partial_\alpha}\vec{e}_x + \frac{\partial y}{\partial_\alpha}\vec{e}_y + \frac{\partial z}{\partial_\alpha}\vec{e}_z \\
\vec{w}_\alpha &= -r\sin\alpha\sin\beta\vec{e}_x + r\cos\alpha\sin\beta\vec{e}_y \\
\left|\vec{w}_\alpha\right| &= \sqrt{r^2 \sin\alpha^2 \sin\beta^2 + r^2 \cos\alpha^2 \sin\beta^2} \\
\left|\vec{w}_\alpha\right| &= r\sin\beta \\
\vec{e}_\alpha &= -\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y
\end{aligned}$$

$$
\begin{aligned}
\vec{w}_\beta &= \frac{\partial x}{\partial_\beta}\vec{e}_x + \frac{\partial y}{\partial_\beta}\vec{e}_y + \frac{\partial z}{\partial_\beta}\vec{e}_z \\
\vec{w}_\beta &= r\cos\alpha\cos\beta\vec{e}_x + r\sin\alpha\cos\beta\vec{e}_y - r\sin\beta\vec{e}_z \\
|\vec{w}_\beta| &= \sqrt{r^2\cos\alpha^2\cos\beta^2 + r^2\sin\alpha^2\cos\beta^2 + r^2\sin\beta^2} \\
|\vec{w}_\beta| &= r \\
\vec{e}_\beta &= \cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z
\end{aligned}
$$

$$
\begin{aligned}
\vec{w}_r &= \frac{\partial x}{\partial_r}\vec{e}_x + \frac{\partial y}{\partial_r}\vec{e}_y + \frac{\partial z}{\partial_r}\vec{e}_z \\
\vec{w}_r &= \cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z \\
|\vec{w}_r| &= \sqrt{\cos\alpha^2\sin\beta^2 + \sin\alpha^2\sin\beta^2 + \cos\beta^2} \\
|\vec{w}_r| &= 1 \\
\vec{e}_r &= \cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z
\end{aligned}
$$

Assembling these results into Eq. 6 gives

$$
\begin{aligned}
\nabla &= \frac{-r\sin\alpha\sin\beta\vec{e}_x + r\cos\alpha\sin\beta\vec{e}_y}{r^2\sin\beta^2}\partial_\alpha + \frac{r\cos\alpha\cos\beta\vec{e}_x + r\sin\alpha\cos\beta\vec{e}_y - r\sin\beta\vec{e}_z}{r^2}\partial_\beta \\
&\quad + (\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z)\,\partial_{w_j} \\
\nabla &= \frac{-\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y}{r\sin\beta}\partial_\alpha + \frac{\cos\alpha\cos\beta\vec{e}_x + \sin\alpha\cos\beta\vec{e}_y - \sin\beta\vec{e}_z}{r}\partial_\beta \\
&\quad + (\cos\alpha\sin\beta\vec{e}_x + \sin\alpha\sin\beta\vec{e}_y + \cos\beta\vec{e}_z)\,\partial_{w_j}
\end{aligned}
$$

$$
\boxed{\nabla = \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha + \frac{1}{r}\vec{e}_\beta\partial_\beta + \vec{e}_r\partial_r} \qquad \text{Gradient in spherical coordinates}
$$

This corresponds to Eq. **??** in **Gradient in Kugelkoordinaten** if we apply this operator to a scalar field. What happens if we apply this operator to a vector field expressed in spherical coordinates?

$$
\vec{A} = A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r
$$

$$
\begin{aligned}
\nabla\vec{A} &= \left(\frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha + \frac{1}{r}\vec{e}_\beta\partial_\beta + \vec{e}_r\partial_r\right)\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) + \frac{1}{r}\vec{e}_\beta\partial_\beta\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
&\quad +\vec{e}_r\partial_r\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) + \frac{1}{r}\vec{e}_\beta\partial_\beta\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
&\quad +\vec{e}_r\partial_r\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right)
\end{aligned}
$$

Note that $\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_r$ have to be differentiated as well.

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\partial_\alpha\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
&\quad \frac{1}{r}\vec{e}_\beta\partial_\beta\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right) \\
&\quad \vec{e}_r\partial_r\left(A_\alpha\vec{e}_\alpha + A_\beta\vec{e}_\beta + A_r\vec{e}_r\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\vec{e}_\alpha\left(\frac{\partial(A_\alpha)}{\partial\alpha}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} + \frac{\partial(A_\beta)}{\partial\alpha}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial\alpha} + \frac{\partial(A_r)}{\partial\alpha}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial\alpha}\right) \\
&\quad +\frac{1}{r}\vec{e}_\beta\left(\frac{\partial(A_\alpha)}{\partial\beta}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\beta} + \frac{\partial(A_\beta)}{\partial\beta}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial\beta} + \frac{\partial(A_r)}{\partial\beta}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial\beta}\right) \\
&\quad +\vec{e}_r\left(\frac{\partial(A_\alpha)}{\partial r}\vec{e}_\alpha + A_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial r} + \frac{\partial(A_\beta)}{\partial r}\vec{e}_\beta + A_\beta\frac{\partial(\vec{e}_\beta)}{\partial r} + \frac{\partial(A_r)}{\partial r}\vec{e}_r + A_r\frac{\partial(\vec{e}_r)}{\partial r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial(A_\alpha)}{\partial\alpha} + A_\alpha\vec{e}_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} + \frac{\partial(A_\beta)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta + A_\beta\vec{e}_\alpha\frac{\partial(\vec{e}_\beta)}{\partial\alpha} + \frac{\partial(A_r)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r + A_r\vec{e}_\alpha\frac{\partial(\vec{e}_r)}{\partial\alpha}\right) \\
&\quad +\frac{1}{r}\left(\frac{\partial(A_\alpha)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha + A_\alpha\vec{e}_\beta\frac{\partial(\vec{e}_\alpha)}{\partial\beta} + \frac{\partial(A_\beta)}{\partial\beta} + A_\beta\vec{e}_\beta\frac{\partial(\vec{e}_\beta)}{\partial\beta} + \frac{\partial(A_r)}{\partial\beta}\vec{e}_\beta\vec{e}_r + A_r\vec{e}_\beta\frac{\partial(\vec{e}_r)}{\partial\beta}\right) \\
&\quad +\frac{\partial(A_\alpha)}{\partial r}\vec{e}_r\vec{e}_\alpha + A_\alpha\vec{e}_r\frac{\partial(\vec{e}_\alpha)}{\partial r} + \frac{\partial(A_\beta)}{\partial r}\vec{e}_r\vec{e}_\beta + A_\beta\vec{e}_r\frac{\partial(\vec{e}_\beta)}{\partial r} + \frac{\partial(A_r)}{\partial r} + A_r\vec{e}_r\frac{\partial(\vec{e}_r)}{\partial r}
\end{aligned}
$$

We have

$$
\begin{aligned}
\vec{e}_\alpha\vec{e}_\alpha &= \vec{e}_\alpha\cdot\vec{e}_\alpha = 1 \\
\frac{\partial\left(\vec{e}_\alpha^2\right)}{\partial\alpha} &= 2\vec{e}_\alpha\frac{\partial(\vec{e}_\alpha)}{\partial\alpha} = 0
\end{aligned}
$$

and thus

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+A_\beta\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha}+\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha}\right)\\
&+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+A_\alpha\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta}+\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+A_r\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta}\right)\\
&+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+A_\alpha\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r}+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+A_\beta\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r}+\frac{\partial\left(A_r\right)}{\partial r}
\end{aligned}
$$

Let's consider the last term in the first row.

$$
\begin{aligned}
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\sin\beta\vec{e}_x+\sin\alpha\sin\beta\vec{e}_y+\cos\beta\vec{e}_z\right)}{\partial\alpha}\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)\sin\beta\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \vec{e}_\alpha^2\sin\beta\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_r\right)}{\partial\alpha} &= \sin\beta
\end{aligned}
$$

Let's consider the last term in the second row.

$$
\begin{aligned}
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta\frac{\partial\left(\cos\alpha\sin\beta\vec{e}_x+\sin\alpha\sin\beta\vec{e}_y+\cos\beta\vec{e}_z\right)}{\partial\beta}\\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta\left(\cos\alpha\cos\beta\vec{e}_x+\sin\alpha\cos\beta\vec{e}_y-\sin\beta\vec{e}_z\right)\\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= \vec{e}_\beta^2\\
\vec{e}_\beta\frac{\partial\left(\vec{e}_r\right)}{\partial\beta} &= 1
\end{aligned}
$$

We look at the fourth term in the last row

$$
\begin{aligned}
\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r} &= \vec{e}_r\frac{\partial\left(\cos\alpha\cos\beta\vec{e}_x+\sin\alpha\cos\beta\vec{e}_y-\sin\beta\vec{e}_z\right)}{\partial r}\\
\vec{e}_r\frac{\partial\left(\vec{e}_\beta\right)}{\partial r} &= 0
\end{aligned}
$$

and the second term in the last row.

$$
\begin{aligned}
\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} &= \vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r}\\
\vec{e}_r\frac{\partial\left(\vec{e}_\alpha\right)}{\partial r} &= 0
\end{aligned}
$$

This gives

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+A_\beta\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha}+\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\sin\beta\right)\\
&+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+A_\alpha\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta}+\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+A_r\right)\\
&+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+\frac{\partial\left(A_r\right)}{\partial r}
\end{aligned}$$

We check out the third term in the first row.

$$\begin{aligned}
\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\cos\beta\vec{e}_x+\sin\alpha\cos\beta\vec{e}_y-\sin\beta\vec{e}_z\right)}{\partial\alpha}\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\cos\beta\vec{e}_x+\cos\alpha\cos\beta\vec{e}_y\right)\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)\cos\beta\\
\vec{e}_\alpha\frac{\partial\left(\vec{e}_\beta\right)}{\partial\alpha} &= \cos\beta
\end{aligned}$$

We check out the second term in the second row.

$$\begin{aligned}
\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta} &= \vec{e}_\beta\frac{\partial\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)}{\partial\beta}\\
\vec{e}_\beta\frac{\partial\left(\vec{e}_\alpha\right)}{\partial\beta} &= 0
\end{aligned}$$

This gives

$$\begin{aligned}
\nabla\vec{A} = &\frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+A_\beta\cos\beta+\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\sin\beta\right)\\
&+\frac{1}{r}\left(\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+A_r\right)\\
&+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+\frac{\partial\left(A_r\right)}{\partial r}
\end{aligned}$$

and after rearraging the terms

$$
\begin{aligned}
\nabla\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+\frac{1}{r\sin\beta}A_\beta\cos\beta+\frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r \\
&\quad+\frac{1}{r\sin\beta}A_r\sin\beta+\frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+\frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r \\
&\quad\quad+\frac{1}{r}A_r+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta+\frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{1}{r\sin\beta}A_\beta\cos\beta+\frac{1}{r\sin\beta}A_r\sin\beta+\frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta} \\
&\quad+\frac{1}{r}A_r+\frac{\partial\left(A_r\right)}{\partial r}+\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+\frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r \\
&\quad\quad+\frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha+\frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta
\end{aligned}
$$

This result has a scalar component and a bi-vector component.

$$\nabla\vec{A}=\nabla\cdot\vec{A}+\nabla\wedge\vec{A}$$

We consider the scalar component first.

$$
\begin{aligned}
\nabla\cdot\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{1}{r\sin\beta}A_\beta\cos\beta+\frac{2}{r}A_r+\frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\cdot\vec{A} &= \frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}+\frac{1}{r\sin\beta}A_\beta\cos\beta+\frac{2}{r}A_r+\frac{1}{r}\frac{\partial\left(A_\beta\right)}{\partial\beta}+\frac{\partial\left(A_r\right)}{\partial r} \\
\nabla\cdot\vec{A} &= \left(\frac{\partial\left(A_r\right)}{\partial r}+2\frac{1}{r}A_r\right)+\left(\frac{\partial\left(A_\beta\right)}{\partial\beta}\frac{1}{r\sin\beta}\sin\beta+A_\beta\frac{1}{r\sin\beta}\cos\beta\right)+\frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha} \\
\nabla\cdot\vec{A} &= \frac{1}{r^2}\left(\frac{\partial\left(A_r\right)}{\partial r}r^2+2A_r r\right)+\frac{1}{r\sin\beta}\left(\frac{\partial\left(A_\beta\right)}{\partial\beta}\sin\beta+A_\beta\cos\beta\right)+\frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}
\end{aligned}
$$

$$\boxed{\nabla\cdot\vec{A}=\frac{1}{r^2}\frac{\partial\left(A_r r^2\right)}{\partial r}+\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\sin\beta\right)}{\partial\beta}+\frac{1}{r\sin\beta}\frac{\partial\left(A_\alpha\right)}{\partial\alpha}}$$

Divergence of a vector field

We also have a bi-vecor component.

$$
\begin{aligned}
\nabla\wedge\vec{A} = &\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_\beta+\frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+\frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\vec{e}_\beta\vec{e}_\alpha \\
&+\frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}\vec{e}_\beta\vec{e}_r+\frac{\partial\left(A_\alpha\right)}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial\left(A_\beta\right)}{\partial r}\vec{e}_r\vec{e}_\beta
\end{aligned}
$$

$$\boxed{\nabla\wedge\vec{A}=\left(\frac{1}{r\sin\beta}\frac{\partial\left(A_\beta\right)}{\partial\alpha}-\frac{1}{r}\frac{\partial\left(A_\alpha\right)}{\partial\beta}\right)\vec{e}_\alpha\vec{e}_\beta+\left(\frac{1}{r}\frac{\partial\left(A_r\right)}{\partial\beta}-\frac{\partial\left(A_\beta\right)}{\partial r}\right)\vec{e}_\beta\vec{e}_r+\left(\frac{\partial\left(A_\alpha\right)}{\partial r}-\frac{1}{r\sin\beta}\frac{\partial\left(A_r\right)}{\partial\alpha}\right)\vec{e}_r\vec{e}_\alpha}$$

2 Cylindrical Coordinates

Let a curvilinear coordinate system be defined by the expression

$$\vec{r}\,(\alpha, \beta, r) = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ z \end{pmatrix}$$

Our aim is to calculate Eq. 6 for this coordinate system ($w_1 = \alpha$, $w_2 = \beta$, $w_3 = r$). We first determine the three

$$\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad\qquad \vec{e}_{w_j} = \frac{1}{|\vec{w}_j|}\vec{w}_j$$

and their magnitudes.

$$\begin{aligned}
\vec{w}_\alpha &= \frac{\partial x}{\partial \alpha}\vec{e}_x + \frac{\partial y}{\partial \alpha}\vec{e}_y + \frac{\partial z}{\partial \alpha}\vec{e}_z \\
\vec{w}_\alpha &= -r\sin\alpha\vec{e}_x + r\cos\alpha\vec{e}_y \\
|\vec{w}_\alpha| &= r \\
\vec{e}_\alpha &= -\sin\alpha\vec{e}_x + \cos\alpha\vec{e}_y
\end{aligned}$$

$$\begin{aligned}
\vec{w}_z &= \frac{\partial x}{\partial z}\vec{e}_x + \frac{\partial y}{\partial z}\vec{e}_y + \frac{\partial z}{\partial z}\vec{e}_z \\
\vec{w}_z &= \vec{e}_z \\
|\vec{w}_z| &= 1 \\
\vec{e}_z &= \vec{e}_z
\end{aligned}$$

$$\begin{aligned}
\vec{w}_r &= \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z \\
\vec{w}_r &= \cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y \\
|\vec{w}_r| &= 1 \\
\vec{e}_r &= \cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y
\end{aligned}$$

Assembling these results into Eq. 6 gives

$$\begin{aligned}
\nabla &= \frac{1}{|\vec{w}_j|^2}\vec{w}_j\partial_{w_j} \\
\nabla &= \frac{1}{r^2}\left(-r\sin\alpha\vec{e}_x + r\cos\alpha\vec{e}_y\right)\partial_\alpha + \vec{e}_z\partial_z + \left(\cos\alpha\vec{e}_x + \sin\alpha\vec{e}_y\right)\partial_r
\end{aligned}$$

$$\boxed{\nabla = \frac{1}{r}\vec{e}_\alpha \partial_\alpha + \vec{e}_z \partial_z + \vec{e}_r \partial_r} \qquad \text{Gradient in cylindrical coordinates}$$

This corresponds to LINK in LINK if we apply this operator to a scalar field. What happens if we apply this operator to a vector field expressed in cylindrical coordinates?

$$\vec{A} = A_\alpha \vec{e}_\alpha + A_z \vec{e}_z + A_r \vec{e}_r$$

$$\begin{aligned}
\nabla \vec{A} &= \left(\frac{1}{r}\vec{e}_\alpha \partial_\alpha + \vec{e}_z \partial_z + \vec{e}_r \partial_r\right)\left(A_\alpha \vec{e}_\alpha + A_z \vec{e}_z + A_r \vec{e}_r\right) \\
\nabla \vec{A} &= \frac{1}{r}\vec{e}_\alpha \partial_\alpha \left(A_\alpha \vec{e}_\alpha + A_z \vec{e}_z + A_r \vec{e}_r\right) + \vec{e}_z \partial_z \left(A_\alpha \vec{e}_\alpha + A_z \vec{e}_z + A_r \vec{e}_r\right) + \vec{e}_r \partial_r \left(A_\alpha \vec{e}_\alpha + A_z \vec{e}_z + A_r \vec{e}_r\right)
\end{aligned}$$

$$\begin{aligned}
\nabla \vec{A} &= \frac{1}{r}\vec{e}_\alpha \left(\frac{\partial}{\partial \alpha}\left(A_\alpha \vec{e}_\alpha\right) + \frac{\partial}{\partial \alpha}\left(A_z \vec{e}_z\right) + \frac{\partial}{\partial \alpha}\left(A_r \vec{e}_r\right)\right) + \vec{e}_z \left(\frac{\partial}{\partial z}\left(A_\alpha \vec{e}_\alpha\right) + \frac{\partial}{\partial z}\left(A_z \vec{e}_z\right) + \frac{\partial}{\partial z}\left(A_r \vec{e}_r\right)\right) \\
&+ \vec{e}_r \left(\frac{\partial}{\partial r}\left(A_\alpha \vec{e}_\alpha\right) + \frac{\partial}{\partial r}\left(A_z \vec{e}_z\right) + \frac{\partial}{\partial r}\left(A_r \vec{e}_r\right)\right)
\end{aligned}$$

$$\begin{aligned}
\nabla \vec{A} &= \frac{1}{r}\vec{e}_\alpha \left(\frac{\partial A_\alpha}{\partial \alpha}\vec{e}_\alpha + A_\alpha \frac{\partial \vec{e}_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial \alpha}\vec{e}_z + A_z \frac{\partial \vec{e}_z}{\partial \alpha} + \frac{\partial A_r}{\partial \alpha}\vec{e}_r + A_r \frac{\partial \vec{e}_r}{\partial \alpha}\right) \\
&+ \vec{e}_z \left(\frac{\partial A_\alpha}{\partial z}\vec{e}_\alpha + A_\alpha \frac{\partial \vec{e}_\alpha}{\partial z} + \frac{\partial A_z}{\partial z}\vec{e}_z + A_z \frac{\partial \vec{e}_z}{\partial z} + \frac{\partial A_r}{\partial z}\vec{e}_r + A_r \frac{\partial \vec{e}_r}{\partial z}\right) \\
&+ \vec{e}_r \left(\frac{\partial A_\alpha}{\partial r}\vec{e}_\alpha + A_\alpha \frac{\partial \vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_z + A_z \frac{\partial \vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r}\vec{e}_r + A_r \frac{\partial \vec{e}_r}{\partial r}\right) \\
&+ \vec{e}_r \left(\frac{\partial A_\alpha}{\partial r}\vec{e}_\alpha + A_\alpha \frac{\partial \vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_z + A_z \frac{\partial \vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r}\vec{e}_r + A_r \frac{\partial \vec{e}_r}{\partial r}\right)
\end{aligned}$$

$$\begin{aligned}
\nabla \vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial \alpha} + A_\alpha \vec{e}_\alpha \frac{\partial \vec{e}_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial \alpha}\vec{e}_\alpha \vec{e}_z + A_z \vec{e}_\alpha \frac{\partial \vec{e}_z}{\partial \alpha} + \frac{\partial A_r}{\partial \alpha}\vec{e}_\alpha \vec{e}_r + A_r \vec{e}_\alpha \frac{\partial \vec{e}_r}{\partial \alpha}\right) \\
&+ \frac{\partial A_\alpha}{\partial z}\vec{e}_z \vec{e}_\alpha + A_\alpha \vec{e}_z \frac{\partial \vec{e}_\alpha}{\partial z} + \frac{\partial A_z}{\partial z} + A_z \vec{e}_z \frac{\partial \vec{e}_z}{\partial z} + \frac{\partial A_r}{\partial z}\vec{e}_z \vec{e}_r + A_r \vec{e}_z \frac{\partial \vec{e}_r}{\partial z} \\
&+ \frac{\partial A_\alpha}{\partial r}\vec{e}_r \vec{e}_\alpha + A_\alpha \vec{e}_r \frac{\partial \vec{e}_\alpha}{\partial r} + \frac{\partial A_z}{\partial r}\vec{e}_r \vec{e}_z + A_z \vec{e}_r \frac{\partial \vec{e}_z}{\partial r} + \frac{\partial A_r}{\partial r} + A_r \vec{e}_r \frac{\partial \vec{e}_r}{\partial r}
\end{aligned}$$

We have

$$\begin{aligned}
\vec{e}_\alpha \vec{e}_\alpha &= \vec{e}_\alpha \cdot \vec{e}_\alpha = 1 \\
\frac{\partial\left(\vec{e}_\alpha^2\right)}{\partial \alpha} &= 2\vec{e}_\alpha \frac{\partial\left(\vec{e}_\alpha\right)}{\partial \alpha} = 0
\end{aligned}$$

and thus

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial\alpha}+\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+A_z\vec{e}_\alpha\frac{\partial\vec{e}_z}{\partial\alpha}+\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha}\right)\\
&+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+A_\alpha\vec{e}_z\frac{\partial\vec{e}_\alpha}{\partial z}+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+A_r\vec{e}_z\frac{\partial\vec{e}_r}{\partial z}\\
&+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+A_\alpha\vec{e}_r\frac{\partial\vec{e}_\alpha}{\partial r}+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+A_z\vec{e}_r\frac{\partial\vec{e}_z}{\partial r}+\frac{\partial A_r}{\partial r}
\end{aligned}$$

Let's consider the last term in the first row.

$$\begin{aligned}
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= \vec{e}_\alpha\frac{\partial\left(\cos\alpha\vec{e}_x+\sin\alpha\vec{e}_y\right)}{\partial\alpha}\\
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= \vec{e}_\alpha\left(-\sin\alpha\vec{e}_x+\cos\alpha\vec{e}_y\right)\\
\vec{e}_\alpha\frac{\partial\vec{e}_r}{\partial\alpha} &= 1
\end{aligned}$$

This gives

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\left(\frac{\partial A_\alpha}{\partial\alpha}+\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+A_r\right)\\
&\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r\\
&\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+\frac{\partial A_r}{\partial r}
\end{aligned}$$

and thus

$$\begin{aligned}
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{1}{r}\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+\frac{1}{r}A_r+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha+\frac{\partial A_z}{\partial z}\\
&+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z+\frac{\partial A_r}{\partial r}\\
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}A_r+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial r}+\frac{1}{r}\frac{\partial A_z}{\partial\alpha}\vec{e}_\alpha\vec{e}_z+\frac{\partial A_\alpha}{\partial z}\vec{e}_z\vec{e}_\alpha\\
&+\frac{1}{r}\frac{\partial A_r}{\partial\alpha}\vec{e}_\alpha\vec{e}_r+\frac{\partial A_\alpha}{\partial r}\vec{e}_r\vec{e}_\alpha+\frac{\partial A_r}{\partial z}\vec{e}_z\vec{e}_r+\frac{\partial A_z}{\partial r}\vec{e}_r\vec{e}_z\\
\nabla\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha}+\frac{1}{r}A_r+\frac{\partial A_z}{\partial z}+\frac{\partial A_r}{\partial r}+\left(\frac{1}{r}\frac{\partial A_z}{\partial\alpha}-\frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z+\left(\frac{1}{r}\frac{\partial A_r}{\partial\alpha}-\frac{\partial A_\alpha}{\partial r}\right)\vec{e}_\alpha\vec{e}_r\\
&+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r
\end{aligned}$$

This result has a scalar component and a bi-vector component.

$$\nabla\vec{A} = \nabla\cdot\vec{A} + \nabla\wedge\vec{A}$$

We consider the scalar component first.

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r}} \qquad \text{Divergence of a vector field}$$

This corresponds to what we have found in Eq. **??**:

$$\begin{aligned}
\operatorname{div}\vec{A} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(A_r r) + \frac{\partial}{\partial\varphi}A_\alpha + \frac{\partial}{\partial z}(A_z r)\right)\\
\operatorname{div}\vec{A} &= \frac{1}{r}\left(\frac{\partial A_r}{\partial r}r + A_r + \frac{\partial A_\alpha}{\partial\alpha} + \frac{\partial A_z}{\partial z}r + A_z\frac{\partial r}{\partial z}\right)\\
\operatorname{div}\vec{A} &= \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha} + \frac{\partial A_z}{\partial z} + \frac{1}{r}A_z\frac{\partial r}{\partial z}\\
\operatorname{div}\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r} + \frac{1}{r}A_z\frac{\partial r}{\partial z}\\
\operatorname{div}\vec{A} &= \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha} + \frac{1}{r}A_r + \frac{\partial A_z}{\partial z} + \frac{\partial A_r}{\partial r}
\end{aligned}$$

We also have a bi-vecor component.

$$\boxed{\nabla\wedge\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \left(\frac{1}{r}\frac{\partial A_r}{\partial\alpha} - \frac{\partial A_\alpha}{\partial r}\right)\vec{e}_\alpha\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r \quad (8)} \qquad \text{Curl of a vector field}$$

This corresponds to what we have found in Eq. **??**.

$$\operatorname{rot}\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\varphi + \frac{1}{r}\left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right)\vec{e}_z$$

$$\nabla \wedge \vec{A} = \text{rot}\,\vec{A}^{*}$$

$$\nabla \wedge \vec{A} = \left(\left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\varphi + \frac{1}{r}\left(\frac{\partial (rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\vec{e}_z\right)\vec{e}_r\vec{e}_\varphi\vec{e}_z$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_r\vec{e}_r\vec{e}_\varphi\vec{e}_z + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\varphi\vec{e}_r\vec{e}_\varphi\vec{e}_z + \frac{1}{r}\left(\frac{\partial (rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\vec{e}_z\vec{e}_r\vec{e}_\varphi\vec{e}_z$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_\varphi\vec{e}_z + \left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right)\vec{e}_r\vec{e}_z + \frac{1}{r}\left(\frac{\partial (rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\vec{e}_r\vec{e}_\varphi$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right)\vec{e}_r\vec{e}_z + \frac{1}{r}\left(\frac{\partial (rA_\alpha)}{\partial r} - \frac{\partial A_r}{\partial \alpha}\right)\vec{e}_r\vec{e}_\alpha$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \frac{1}{r}\left(\frac{\partial (rA_\alpha)}{\partial r} - \frac{\partial A_r}{\partial \alpha}\right)\vec{e}_r\vec{e}_\alpha + \left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right)\vec{e}_r\vec{e}_z$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \frac{1}{r}\left(\frac{\partial A_r}{\partial \alpha} - \frac{\partial (rA_\alpha)}{\partial r}\right)\vec{e}_\alpha\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r$$

$$\nabla \wedge \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{e}_\alpha\vec{e}_z + \frac{1}{r}\left(\frac{\partial A_r}{\partial \alpha} - \left(A_\alpha + r\frac{\partial A_\alpha}{\partial r}\right)\right)\vec{e}_\alpha\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_z\vec{e}_r$$

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This is not 100% compatible with Eq. 8. We have to revisit this.

3 Reciprocal bases

Resource: https://arxiv.org/pdf/2206.02459.pdf A coordinate system is orthogonal if the basis vectors $\{\vec{w}_j\}$ are orthogonal. Considerable simplifiation then occurs, because the reciprocal basis vectors $\vec{w}^j$ can then easily be computed (Eq. 4).

$$\nabla = \vec{w}^j\partial_{w_j} = \frac{1}{|\vec{w}_j|^2}\vec{w}_j\partial_{w_j} = \frac{1}{|\vec{w}_j|}\vec{e}_{w_j}\partial_{w_j}$$

Let's consinder the general case of non-orthogonal basis vectors now. From definitions Eq. 1 and Eq. 2

$$\vec{w}_k = \frac{\partial r_i}{\partial w_k}\vec{e}_i \qquad \vec{w}^j = \frac{\partial w_j}{\partial r_i}\vec{e}_i$$

we got (see Eq. 3)

$$\vec{w}_k \cdot \vec{w}^j = \frac{\partial w_j}{\partial w_k} = \begin{cases} 1 & : \quad for j = k \\ 0 & : \quad for j \neq k \end{cases}$$

so for a given index i we have

$$\vec{w}_i \cdot \vec{w}^i = 1 \qquad \text{(no sum)}$$

The vectors $\vec{w}_i$ and $\vec{w}^i$ do not necessarily have to be parallel for the projection $\vec{w}_i \cdot \vec{w}^i$ to be 1.

Let G be an n-dimensional geometric algebra such that $I^2 \neq 0$ (-1 for $n = 3$, 1 for $n = 4$,...). Let $\vec{e}_i$ be a basis for the vectors in this algebra. Then the reciprocal bases $\vec{e}^i$ is defined as

$$\boxed{\vec{e}^i = \left((-1)^{i-1}\left(\vec{e}_1 \wedge \wedge \tilde{\vec{e}}_i \wedge \wedge \vec{e}_n\right)\right) \cdot I^{-1}} \tag{9}$$

in which $\tilde{\vec{e}}_i$ means that this index is omitted from the outer product and in which the dot represents the generalized inner product of two blades, that is the lowest grade part of their geometric product. In **Geometric product of a vector with a bi-vector** we have produced that geometric product of a bibvector with a vector and got a grade-1 component and a grade-3 component. In **Geometric product of a bi-vector with a bi-vector** we multiplied two bivectors and got a grade-0 component and a grade-2 component. The generalized inner product is of grade $n-k$ where n is the grade of the factor with the higher grade and k the grade of the factor with lower grade. In Eq. 9 we produce the inner product of a $n-1$ vector with an n - vector. This yields a result of grade-1.

Theorem: If $\vec{e}_i$ is an orthonormal basis then $\vec{e}_i = \vec{e}^i$.

Proof: We have

$$\vec{e}^i = \left((-1)^{i-1}\left(\vec{e}_1 \wedge \wedge \tilde{\vec{e}}_i \wedge \wedge \vec{e}_n\right)\right) \cdot I^{-1}$$

Since the base vectors are orthogonal we can convert the outer product into a geometric product. The inner product extracts the lowest grade component.

$$\begin{aligned}
\vec{e}^i &= (-1)^{i-1}\langle\left(\vec{e}_1\tilde{\vec{e}}_i\vec{e}_n\right)(\vec{e}_n\vec{e}_1)\rangle_{lowest} \\
\vec{e}^i &= (-1)^{i-1}\langle\vec{e}_1\tilde{\vec{e}}_i\vec{e}_n\vec{e}_n\vec{e}_1\rangle_{lowest} \\
\vec{e}^i &= (-1)^{i-1}\langle\vec{e}_1\tilde{\vec{e}}_i\vec{e}_{n-1}\vec{e}_{n-1}\vec{e}_1\rangle_{lowest} \\
\vec{e}^i &= (-1)^{i-1}(-1)^{i-1}\langle\vec{e}_i\rangle_{lowest} \\
\vec{e}^i &= (-1)^{i-1}(-1)^{i-1}\vec{e}_i \\
\vec{e}^i &= \vec{e}_i
\end{aligned}$$

One of the main purposes of the reciprocal frame is to *quickly compute* the coordinates of a vector with respect to a *non-orthogonal basis.* In fact, the reciprocal basis is so constructed that if $\vec{b} = b_i\vec{e}_i$ defines a given vector then

$$\boxed{b_i = \vec{e}^i \cdot \vec{b}}$$

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One of the main purposes of the reciprocal frame is to *quickly compute* the coordinates of a vector with respect to a *non-orthogonal basis.* In fact, the reciprocal basis is so constructed that if $\vec{b} = b_i\vec{e}_i$ defines a given vector then

$$\boxed{b_i = \vec{e}^i \cdot \vec{b}}$$

Let

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{w}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be a non-orthonogal basis of R^3. We show that the basis in non-orthonogla by computing

$$\vec{w}_1 \cdot \vec{w}_2 = 4$$

Let's now consider the vector

$$\vec{b} = 2\vec{e}_x + 3\vec{e}_y + 4\vec{e}_z \tag{11}$$

What will be the components of this vector with respect to the basis $\vec{w}_i$? Let's calculate Eq. 10.

$$\begin{aligned}
\vec{w}^1 &= \left((-1)^{1-1}\left(\vec{w}_2 \wedge \vec{w}_3\right)\right) \cdot I^{-1} \\
\vec{w}^2 &= \left((-1)^{2-1}\left(\vec{w}_1 \wedge \vec{w}_3\right)\right) \cdot I^{-1} \\
\vec{w}^3 &= \left((-1)^{3-1}\left(\vec{w}_1 \wedge \vec{w}_2\right)\right) \cdot I^{-1}
\end{aligned}$$

We rewrite the wedge products as geometric products.

$$\begin{aligned}
\vec{w}^1 &= \left(\frac{1}{2}\left(\vec{w}_2\vec{w}_3 - \vec{w}_3\vec{w}_2\right)\right) \cdot I^{-1} \\
\vec{w}^2 &= -\left(\frac{1}{2}\left(\vec{w}_1\vec{w}_3 - \vec{w}_3\vec{w}_1\right)\right) \cdot I^{-1} \\
\vec{w}^3 &= \left(\frac{1}{2}\left(\vec{w}_1\vec{w}_2 - \vec{w}_2\vec{w}_1\right)\right) \cdot I^{-1}
\end{aligned}$$

and do the generalized inner product trick shown above.

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(\vec{w}_2\vec{w}_3 - \vec{w}_3\vec{w}_2\right)\vec{e}_3\vec{e}_2\vec{e}_1\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(\vec{w}_1\vec{w}_3 - \vec{w}_3\vec{w}_1\right)\vec{e}_3\vec{e}_2\vec{e}_1\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(\vec{w}_1\vec{w}_2 - \vec{w}_2\vec{w}_1\right)\vec{e}_3\vec{e}_2\vec{e}_1\rangle_{lowest}
\end{aligned}$$

We need to substitute $\vec{w}_i$ now.

$$\vec{w}_1 = 2\vec{e}_x + \vec{e}_y \qquad \vec{w}_2 = \vec{e}_x + 2\vec{e}_y \qquad \vec{w}_3 = \vec{e}_z$$

This gives

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(\left(\vec{e}_x + 2\vec{e}_y\right)\vec{e}_z - \vec{e}_z\left(\vec{e}_x + 2\vec{e}_y\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(\left(2\vec{e}_x + \vec{e}_y\right)\vec{e}_z - \vec{e}_z\left(2\vec{e}_x + \vec{e}_y\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(\left(2\vec{e}_x + \vec{e}_y\right)\left(\vec{e}_x + 2\vec{e}_y\right) - \left(\vec{e}_x + 2\vec{e}_y\right)\left(2\vec{e}_x + \vec{e}_y\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

and then

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(\vec{e}_x\vec{e}_z + 2\vec{e}_y\vec{e}_z - \left(\vec{e}_z\vec{e}_x + 2\vec{e}_z\vec{e}_y\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(2\vec{e}_x\vec{e}_z + \vec{e}_y\vec{e}_z - \left(2\vec{e}_z\vec{e}_x + \vec{e}_z\vec{e}_y\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(2\left(\vec{e}_x\vec{e}_x + 2\vec{e}_x\vec{e}_y\right) + \left(\vec{e}_y\vec{e}_x + 2\vec{e}_y\vec{e}_y\right) - \left(\left(2\vec{e}_x\vec{e}_x + \vec{e}_x\vec{e}_y\right) + 2\left(2\vec{e}_y\vec{e}_x + \vec{e}_y\vec{e}_y\right)\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(\vec{e}_x\vec{e}_z + 2\vec{e}_y\vec{e}_z - \vec{e}_z\vec{e}_x - 2\vec{e}_z\vec{e}_y\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(2\vec{e}_x\vec{e}_z + \vec{e}_y\vec{e}_z - 2\vec{e}_z\vec{e}_x - \vec{e}_z\vec{e}_y\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(\left(2 + 4\vec{e}_x\vec{e}_y\right) + \vec{e}_y\vec{e}_x + 2 - \left(2 + \vec{e}_x\vec{e}_y\right) - \left(4\vec{e}_y\vec{e}_x + 2\right)\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(2\vec{e}_x\vec{e}_z + 4\vec{e}_y\vec{e}_z\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(4\vec{e}_x\vec{e}_z + 2\vec{e}_y\vec{e}_z\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(2 + 4\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x + 2 - 2 - \vec{e}_x\vec{e}_y - 4\vec{e}_y\vec{e}_x - 2\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle\left(2\vec{e}_x\vec{e}_z + 4\vec{e}_y\vec{e}_z\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle\left(4\vec{e}_x\vec{e}_z + 2\vec{e}_y\vec{e}_z\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle\left(6\vec{e}_x\vec{e}_y\right)\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle 2\vec{e}_x\vec{e}_z\vec{e}_z\vec{e}_y\vec{e}_x + 4\vec{e}_y\vec{e}_z\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle 4\vec{e}_x\vec{e}_z\vec{e}_z\vec{e}_y\vec{e}_x + 2\vec{e}_y\vec{e}_z\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle 6\vec{e}_x\vec{e}_y\vec{e}_z\vec{e}_y\vec{e}_x\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= \frac{1}{2}\langle 2\vec{e}_z\vec{e}_y + 4\vec{e}_x\rangle_{lowest} \\
\vec{w}^2 &= -\frac{1}{2}\langle -4\vec{e}_y + 2\vec{e}_x\rangle_{lowest} \\
\vec{w}^3 &= \frac{1}{2}\langle 6\vec{e}_z\rangle_{lowest}
\end{aligned}$$

$$\begin{aligned}
\vec{w}^1 &= 2\vec{e}_x \\
\vec{w}^2 &= 2\vec{e}_y - \vec{e}_x \\
\vec{w}^3 &= 3\vec{e}_z
\end{aligned}$$

Now that we have calculated the reciprocal basis $\vec{w}^i$ we can calculate the coefficients in $\vec{b} = b_i\vec{w}_i$ like this

$$b_i = \vec{w}^i \cdot \vec{b}$$

for our vector Eq. 11. This gives

$$\begin{aligned}
b_1 &= (2\vec{e}_x) \cdot (2\vec{e}_x + 3\vec{e}_y + 4\vec{e}_z) \\
b_2 &= (2\vec{e}_y - \vec{e}_x) \cdot (2\vec{e}_x + 3\vec{e}_y + 4\vec{e}_z) \\
b_3 &= (3\vec{e}_z) \cdot (2\vec{e}_x + 3\vec{e}_y + 4\vec{e}_z)
\end{aligned}$$

$$\begin{aligned}
b_1 &= 4 \\
b_2 &= 8 \\
b_3 &= 12
\end{aligned}$$

We check this result by calculating $\vec{b} = b_i\vec{w}_i$ with these coefficients.

$$\vec{b} = b_i\vec{w}_i$$

$$\begin{aligned}
\vec{b} &= 4\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + 12\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\vec{b} &= \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 8 \\ 16 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \cdot 2 \end{pmatrix}
\end{aligned}$$

This result is wrong. We should revisit this chapter.